

**Embodied Cognition and the Origins of Geometry:
A Model Approach of Embodied Mathematics
Through Geometric Considerations**

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Abstract

In this paper, we propose that 'embodied mathematics' should be studied not only by reduction to the present individual bodily experience but in an historical context as well, as far as the origins of mathematics are concerned. Some early mathematical results are the Theorems of Geometry and arose as attempts to objectively render the main perceptual categories such as verticality, horizontality, similarity (or its varieties). Inasmuch as these are of a qualitative nature, it was required that they be expressed in a quantitative way in order to be objectified. The first form of this objectification occurred in the case of 'archetypal results', namely the Pythagorean triads and the internal ratio of the legs in the right triangles. In the next stage, a 'scientific' treatment would come from a shift of objectification and descriptions inside an abstract theory, which would constitute the first logicomathematical knowledge. In this theory, the 'archetypal results' were incorporated, generalized and acquired their unquestionable, supertemporal validity. The study presents a particular epistemological analysis of some of the main terms used in the beginnings of Geometrical Thought and Euclid's Elements, utilizing the theoretical apparatus of the theory of 'embodied mathematics'. It also traces models of objectification for the 'archetypal results' and indicates their diffusion in later mathematical developments.

Key words: Archetypal results, conceptual categories, embodied mathematics, objectification, pattern, perceptual categories, prototypes.

1. The embodied situation

The idea of embodied mathematics was suggested recently by Nunez & al. (1999) and Lakoff & Nunez (2000). The relevance of the human body to its conceptual systems is decisive in the theory of embodied mind in general, as a theory in which the correlation between human experience and cognitive sciences is attempted, Varelas & al. (1992). The leading idea in

embodied mathematics is the mediation of spatiality in the formation of logical thought. In this respect, the comprehension of space (or its models) by humans is of primary importance in the history of knowledge and in cognition as well. Lakoff discusses embodied mathematics at least since 1987, when he stated "mathematics is based on structures within the human conceptual system, structures that people used to comprehend ordinary experience" (Lakoff 1987, p. 364) and he expressed the ambition of unraveling the mystery of Platonic ideas (ibid.). Today, modifications of the same idea are also used by researchers in education, Gray & Tall (2000), Edwards (1998), Boero P. & Bazzini L. & Garutti R, (2002), Watson (2002), Watson & Spyrou & Tall (2003), etc. The critique of the above idea by Presmeg (2002) or Schiralli & Sinclair (2003) is of great interest as well. Lakoff & Nunez, in particular, suggest a list of properties of embodied mathematics, supporting the idea that mathematics is a product of the full range of human experience:

Human mathematics is embodied, it is grounded in bodily experience in the world... not purely subjective... not a matter of mere social agreement...

It uses the very limited and constrained resources of human biology and is shaped by the nature of our brains, our bodies, our conceptual systems, and the concerns of human societies and cultures, Lakoff & Nunez (pp 348 -365).

On the other hand, embodied mathematics allows us to address the main epistemological problem of mathematics (Lakoff 1987, pp. 353-369). We prefer the formulation of Piaget[1]:

The epistemology of mathematics has three principal and classic problems: why is it so fruitful though based on very few and relatively poor concepts or axioms; why has it a necessary character, thus remaining constantly rigorous despite its constructive character which could be a source of irrationality; and why does it agree with experience or physical reality in spite its completely deductive nature?, Piaget (1972, p 69).

The origin of such ideas in fact goes back to the phenomenological tradition of Derrida (1962), Merleau-Ponty (1954) or even that of Husserl (1917). A part of this heritage is acceptable by Varelas & al (1992) or Nunez & al (1999) as well. We do not deal with the various interpretations of the notion or the evolution of the embodied cognition inasmuch as it is very well described in the above respective literature. In the case of geometry, we suggest a model of exposition for this line of ideas by using historical references. This point of view seems to be very little discussed and developed so far.

In this regard and in order to describe the embodied situation, we have to reflect on perception, not only in the context of contemporary neuroscience but also in the light of historical epistemology. This genetic approach inquires into the potential environments, the necessary metaphors and ideas that have preceded the scientific formation in mathematics, as we know it today.

It is remarkable that, in Husserl's writings we come across a program for a recursive inquiry into the "origins of geometry", as the basis of our culture [2]. This program intended to discover the ground of the axioms in the pre-scientific period of geometry, since the established symbolism has covered over the origins. However, the human capacity of using and manipulating symbols has become dominant in the evolution of civilization. But, if we deal with geometry, special attention should be given to the fact that the very symbols used in this domain (such as point, line, etc.) must resemble the mental object we seek to symbolize (Beth & Piaget, p. 217). In this sense, we view these correspondences (between geometric

symbols and concepts) as inextricably bound to our bodily experience and we advocate that embodied mathematics can be presented in an historical perspective, especially as far as their origins, which are related to geometry, are concerned.

1.1 Perception and the formation of geometric notions

In order to investigate the embodied primitive concepts, another web of ideas is needed that connects us with our hypostasis as intelligent beings and is inherent in our perceptual bodily experience in a nature dominated by gravity [3]. Note that Merleau-Ponty underlined the significance of gravity in the comprehension of space and his conviction is supported by others, ever since: Zuzne, (1970), Ibbotson & Bryant (1976), Varelas & al (1992) and Lakoff (1987), indicatively. Actually, we need to distinguish the main components of this perceptual system insofar as they seem to be related to geometry. According to Lakoff & Johnson (1980 p. 277) verticality is the main source domain and is connected with our understanding of quantity as well. There is also evidence that the function of apprehension of the vertical has an affinity to the horizontal, as has been proposed in psychology; the acquisition of the vertical is synchronous with that of the horizontal, Piaget & Inhelder (1956), Mackay & al. (1972). Apart from these two perceptual categories, another important aspect of the cognitive apparatus is the perception of Gestalts [4] that gives us the ability to recognize the 'shape' or 'form' of objects.

Subsequently, we will focus on the above three functions that mainly constitute our perceptual apparatus, i.e., the vertical, the horizontal, and the recognition of the shape (: our ability to apprehend the similarity of shapes). Altogether they are connected functionally, decisively affect our adaptation in the environment, and offer us the main mental tools to describe our experiences in the world. We should note here that similarity generally is a far broader perceptual category: it appears either as visual (figures or coloring) or auditory etc, and consists of a general trend both the perceptual and conceptual systems that unify the manifold of experience into rules, Gentner & Medina (1998). Thus the recognition of visual shape appears to be a particular function of the skill of apprehending similarity. If we consider similarity as a quality, it is difficult to communicate since qualities are subjective impressions. Finally, by using numbers to interpret aspects of similarity and to communicate them; thereby, a sort of inter- subjective knowledge is achieved [5].

In this paper we claim something stronger; human bodily experience of the world is transformed to rationality through the mediation of the geometric comprehension of the world [6]. In particular, we claim that the first results of Geometry arose from the persistent effort for an objective rendering and reification of the three main perceptual categories of space forms, via arithmetic and even logical relationships. In their first formulation such results (namely the Pythagorean triads, the internal ratio of the legs in the right triangles, etc) occurred as 'archetypal' formations of later basic theorems of Geometry. On the other hand, along these lines, a reduction of the formation and development of the geometric thought to a psychological background is also suggested.

2. Archetypal results, Patterns and Prototypes in Geometry

Mathematics is founded on the logicomathematical ability of enumeration. Humankind's apprehension of the idea of number was a cornerstone of objectivity. The significance of

numbers increased when numerical systems improved and allowed the enumeration of huge quantities, and made their operations simpler. Through numbers, the experience becomes homogeneous, inter-subjective easier to transmit and as a result, number stands for a certain grasping of the world's truth [7]. Another set of practical questions (connected to Geometry), in which numbers were involved, had to do with the measurement of land areas. Beyond this task, the construction of homes, temples and pyramids evoked deep questions related to the classification of shapes. In order to understand these spatial phenomena and especially their contribution to the formation of geometric concepts Bender and Schreiber [8] (1980) suggested the very idea of norm and argued in favor of linking the connection of daily activities and notions of geometric shapes.

At any rate, using current terms, we suggest another exposition of the above process. In this, a series of brain and linguistic functions intervene in our conceptual systems, mainly of prototypes, a linguistic term introduced by Rosch (1978). In our context a prototype is regarded as the 'best exemplar' of a concept (and is often 'a non-existence'), as a specification for the members of a family and in this sense, it is the ideal core of a concept, Harley (1995, p. 193), Malt [9] (1999, p. 333). Lines, points, and planes, in their ideal use, serve as our main examples for prototypes in Geometry. The prototypes are also involved in the conceptual evolution of student's geometric thought Tall (1995), Tall & al (2000). Tall & al attempted a weak description of the involvement of prototypes in developing cognitive structures and advanced mathematical thinking.

In our study, we suggest that between the primary notions articulated via prototypes and the archetypal geometric results an additional conceptual structure should mediate. This process is carried out by means of a constructed hierarchy referring to notions of growing complexity, for which we propose the term patterns. In this hierarchy first comes the pattern of a triangle, as it involves a minimum of data (point, line and plane) in a coherent structure. (Note that next in this classification could be the tetrahedron, a distinct spatial pattern).

As a pattern, the triangle is the first step for the analysis of a figure and its reduction to prototypes. In a triangle, the broader categories of equality, similarity, and area, bestow a quality of additional structure and bring forth associations (of plane geometry, for example). Thus, the role of a triangle is fundamental and it becomes the principal instrument, mediating in all proofs concerning more complicated figures. According to this point of view the pattern of the triangle is a necessary and decisive element for the later development of Geometry.

3. The reduction of the archetypal results to Prototypes

The archetypal results constituted potentially an objective knowledge, before the emergence of any coherent logical deductive theory (as is also indicated by a position of Lakatos, 1997). In this respect, it is altogether interesting to locate the necessary regressive processes the mind has followed in order to refine for itself the fundamental terms and finally form a logical theory where the archetypal results could be placed. In any case, the meaning and the semantics of abstract geometric thought consist of exactly these recursive functions, which lead to the determination of the minimum and sufficient terms and principles (the axioms) that constitute the theory [10].

A key instrument of investigation of Greek mathematics was focused around the fundamental (platonic) demand of the anti-visual, as Szabo has noticed:

It seems that new kinds of proof appeared at the same time as Greek mathematics was becoming anti-empirical and anti-visual, Arprad Szabo (1978, p 197).

In order for the archetypal results to come under the control of logic and the requirements of the anti-visual, the ideal configuration that determines their possible rendering in a necessary and unique way had to be invented. In Euclidean Geometry for instance, we have an apparent result: "Two straight lines may intersect at one, and only one, point". Such a formulation of a property of geometric objects could only be arrived at after grasping the prototypical notion of straight line [11].

This approach, and the attendant type of reasoning, actually designates a shift in the process of the objectification of knowledge:

But to make an object of something, to make it a subject of a predication or attributions, merely differs in name from having a presentation of it, and having a presentation in a sense which, while not the only one, is none the less the standard one for logic. (Husserl, Logical Investigation II, p. 366)

3.1 Angle: a compound Prototype

The 'angle' is crucially involved in the fundamental perceptual categories, as they are presented in Section 2, inasmuch as it obviously mediates in shape recognition. Also the same prominent role for the angle is reserved by mathematicians in the early foundational processes, and thus it turns out to be an unavoidable term in any kind of reasoning. In Euclid's Elements (I, Definition 8 and 9), the angle is defined as follows:

A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line. And when the lines containing the angle are straight, the angle is called rectilinear. (quoted in Heath, 1956 p. 176).

Thus, at least within the scientific status of the theory, the angle is not given as a measurable magnitude. It rather occurs as a compound prototype, reduced to the simpler basic prototypes of the point and straight line. The compound prototype is anticipated by the theory [12]. The angles are only amenable to the single manipulation of superposition and coincidence, in the same uniform way that occurs for all figures in Euclid's Geometry. These synthetic conditions do not constitute objective criteria justifying the angle, independent of the subjective experience, like those provided by measurement. Besides, two angles are similar if and only if they are congruent. Thus the angle becomes a form which determines the shape [13].

Note that an acceptable mathematical measurement of the angle is not at all obvious and turns out to be a deep result, both in mathematics [14] and in neurology [15]. However, through a sophisticated treatment in Euclid's Elements, the angles are subject to manipulation in an absolute way, without any mediation of measurement or theory.

In the case of perpendicular lines and in parallels as well, either the definitions or their verification is reduced to certifications concerning prototypical configurations, such as point and straight line:

In fact Euclid does not use any relation which is not reducible to coincidability between lines until he treats ratios. (Mueller 1981, p. 41).

In particular: A straight line meets another straight line perpendicularly, when the formed angles are equal - then the angles are called right angles - All right angles are equal, (Elements I, Definition 10 and Postulate 4). In the above statements a twofold purpose is achieved: First, it serves as the identification condition for perpendicularity, via right angles. Further, through the use of 'all' in the above assertion, an implicit transcendental definition is formulated.

Figure 1: Lines l and l are perpendicular iff $a = b \perp$

In the case of parallel lines, their definition includes a transcendental term, such as 'produced indefinitely' (Heath, 1956, p. 190). We remark again that the formulation of the 5th Postulate (ibid. p. 202) permitted a finitistic type of argument, referring again to straight lines (see Figures 1 and 2).

Figure2: Lines l and l are parallel iff $a + b$ would form a straight line

We consider all the above as indicative and exemplary cases of reduction to prototypes. In fact, it is enough to ascertain that after a suitable 'displacement', we arrive at an arrangement that proves to be a straight angle.

3.2 The triangle as a Pattern

As we have already noticed the triangle is introduced not only as a figure but also as a structure, that is followed immediately by a number of connotations, leading to the conditions of identity and difference that will determine its ontology. Subsequently, a triangle - and as a consequence any complicated geometrical object - would be acceptable if and only if we had rigidity criteria for the particular structure that would make it recognizable in all its potential appearances and permissible transformations.

This logical determination of the triangle was set up after the establishment of the 'congruence' and 'similarity' criteria. The criteria contain some potential actions that are advanced in mental acts of comparisons. These universal properties provide a status for the notion of the triangle, for which we propose the term pattern.

The triangle would eventually constitute the touchstone ensuring logical comparisons and arguments. Situations of this kind arise through the introduction of new concepts, as well as appear in the majority of the proofs. Think, for instance, of the result concerning the angles - sum for a triangle and its variations that determines the type of Space Geometry (Euclidean, Spherical or Hyperbolic).

4. From archetypal results to geometrical theorems

4.1 Pythagorean Theorem

The constitution of a theory for Geometry and in particular the need to encompass the already existing 'archetypal' results posed certain methodological problems. This necessitated the further derivation of mental instruments, as the accumulated empirical knowledge was not enough.

Concerning the Pythagorean Theorem, such a new concept is that of a square which denotes the second power of a number and at the same time is apprehended as the area of a rectangle. This situation might be considered as an early twofold representative expression, both of arithmetic and geometric nature, concerning two-dimensional objects. Another cognitive factor arising is the need for formulations and proofs in terms of the theory under development and at the same time obeying Logic. In the case of a square, its existence seems to be altogether not self-evident and determination of its existence actually requires a proof. This, moreover, inaugurates a new ontology that does not identify the signifier with the signified. Thus the square becomes a mental object and it is derived only by logical construction, its approximative representation not being enough. Indeed, the presentation and the proof of the Pythagorean Theorem (as it is given in Euclid's Elements) assumed, besides the configuration of an orthogonal triangle, the pattern of a square and its representations as well (Elements I, 46-48).

The previous statement suggests that the transition from the archetypal formulation to a rigorous geometric interpretation demanded not only a high degree of abstraction, intuition, and invention, but also a successful work on the foundations. Considering the epistemological character of the particular result, we notice that a purely perceptual category such as verticality:

- (i) Has being objectified at an early stage, through arithmetical relations (namely, Pythagorean Triads), and finally,
- (ii) Transformed into a general mathematical form (Pythagorean theorem), obtaining a universal conceptual validity.

4.2 Similarity

Some archetypal results may constitute the first efforts of the determination of the shape by means of numerical relationships. In particular, archetypal results concerning right angles evolved into the Pythagorean Theorem. On the other hand, there is evidence that attempts for analogous numerical expressions, not involving in certain cases only right angles appeared in ancient Egyptian texts, under the name of "se-qet" (Heath, 1975, pp. 126 - 128). We should note that, the geometric significance of these two results rested upon a main feature of the shape, namely the angle.

Furthermore, the plotting of a figure under scale usually preserves angles. Primal mathematical activities, which traditionally are attributed to Thales, were realized on the ground of angle invariance and they were obviously related to the similarity of figures, as we mean it today. The similarity relationship classifies the figures and distinguishes the shape, apart from the magnitude. This leads to wider and directly "readable" classes of objects. The theory of similarity constitutes the laying down of the criteria that explained and established the invariance of the visual form. A necessary premise for this process is the shift from relations concerning two figures to internal relations that refer to one and the same figure.

We should stress that the objectification of the notion of similarity would not be achieved without the development of the Eudoxian theory of proportions (or, equivalently, Dendekid's theory of real numbers). In the exceptional case of the regular polygons the similarity is self-evident: given n (a natural number), any two regular n -polygons are similar. As we know today, such a powerful situation is due to structural invariance resulting from symmetry

requirements for plane figures.

Another remarkable case of automatic similarity is traced back to the writings of Plato and his intention to describe the rigidity of the forms:

...rectilinear surface is composed of triangles, and all triangles are originally of two kinds.... Both of which are made up of one right and two acute angles....

One of them has at either end of the base half of the divided right angle, having equal sides, while in the other the right angle is divided into unequal parts, having unequal sides...

Now of the two triangles, the isosceles has one form only; the scalene or unequal - sided has an infinite number...

Of the infinite forms we must again select the most beautiful... Then let us choose two triangles... one isosceles, the other having the square of the longer side equal three times the square of the lesser side, (Plato, *Timaeus*, 53d-54b).

In both the above cases, Plato achieves rigidity through the internal ratio of two sides. In the next, he refers to the figures derived from the division of an equilateral triangle by its height i.e., orthogonal triangles with angles 30 and 60 degrees. He also deals with isosceles orthogonal triangles created when drawing the diagonal of a square [16].

Considering two triangles, their angle equality is a consequence of their side proportion and vice-versa, a fact which definitely fails for other polygons. As a matter of fact, the apprehension of the general similarity definition, as it appears (in a complete form) in Euclid's *Elements* [VI, Definitions 1 and 2], deserves a careful historical investigation. We know that in evaluating such an attempt, inherent epistemological problems are raised, which in Euclid's foundation are hidden behind the formulation of the 5th postulate [17].

Since the definition of similarity requires the equality of ratios, its testing demands the potential infinite. Therefore, in the case of similarity criteria we have an inherent, non-finitistic, and simultaneously transcendental character. It is a remarkable fact, that if we concentrate on the fundamental pattern of the triangle then the angle's equality is sufficient for a decision on the similarity of the triangles! After this, the triangle has been established as the basic methodological tool for the proof procedure in synthetic geometry. The transcendental character of similarity was implicit in the manipulations using triangles.

5. Diffusion

Archetypal results are transformed, modified, and their variants finally are greatly dispersed in quite different theories. The manifold of features that allowed changes is related to the intension of the result. Often, the context itself can be a factor of change leading to the transformation of the original result and its re-emergence in another theory. In the rest of our study, for this process we utilize the term diffusion. Insofar as archetypal results constitute aspects of objectification of the perceptual categories, the proposed approach has obviously an epistemological and cognitive character. On the other hand, their reactivation in scientific programs and proposals demands a purely mathematical and historical research. Our purpose is to carry out parts of the above project related to the reactivation of the Pythagorean Theorem, focusing on certain diffusions.

In this respect, there is an early direct generalization [Euclid's *Elements*, VI.31], which re-

places squares with arbitrary similar polygons drawn on the sides of an orthogonal triangle and proves that the same relation concerning their areas is also true. We have in this case the change of single element of the original result, in particular the squares, while the rest of the context is left invariant. Another modification is obtained after the variation of the right angle to an acute or obtuse angle [Elements, II.12 -13], where a well-known generalization is again proved. Along the same lines we can find corresponding results about areas and volumes of 'polytopes' in space [18].

In the Cartesian approach to Geometry, we read the Pythagorean Theorem as a result involving magnitudes, representing directly the sides of the orthogonal triangle: the knowledge of two of them allows the calculation of the third, $a = \sqrt{b^2 + c^2}$. This formulation involves not directly the geometric objects, but the mediation of an algebraic representation for them and actually constitutes in a conceptual shift. We recognize that the obtained relation is not given in terms of 'first reference' (Klein, 1981, p, 28). Thus, the Pythagorean Theorem becomes now fundamental, as a calculating tool involving lengths of line segments, in terms of coordinates. Furthermore, this establishes the quadratic form for the metric, a fact of paramount importance for modern mathematics.

In this modern perspective, a new insight about space conception will lead to non-Euclidean Geometry. In such deviating contexts, exact formulations (like Hyperbolic Trigonometry) are again available, which approximately resemble the original Pythagorean Theorem. At this stage of scientific evolution, an epistemological evaluation of the original result is possible. It is not surprising, inasmuch as we know today that this theorem is in fact equivalent to the 5th Postulate and to the Theory of Similarity as well [19]. Thus the incorporation and the proof of an archetypal result into a theory, probably acts as a pivot that also dominates the constitution of the whole theory.

After the development of the Infinitesimal Concepts and Calculus, 'approximation' methods were adopted by Geometry. Differential Geometry realizes a distinction upon notions of local or global as fields of investigation. In this setting, the metric character inherited from the Pythagorean Theorem has survived, although the Euclidean situation refers to the first steps of an approximating procedure, determining the local [20]. However, variations of the original Pythagorean result can be traced in almost every modern mathematical theory, even if they are not directly related to the classical views of Geometry. Along these lines, Kandisson (2002) presented interesting implications for the extension and utilization of the Pythagorean Theorem.

All the above suggest a diffusion of archetypal results in the total corpus of Mathematics. At the same time, they constitute a confirmation of the objectification process of our basic perceptual categories and the reactivation of the idea of embodiment in the more abstract mathematical fields.

6. Conclusions

The main goal of our study is to make a contribution to the theory of embodied mathematics and its epistemological consequences. The main view of the authors is guided by the interpretations of the phenomenological tradition. In the Epistemology of Mathematics, this imposes the enlargement of our horizons in order to include human experience and its inherent possibility for transformations (Varelas & al). The very mechanism is based on

experience, but could not be reduced solely to a psychological basis. According to Vygotsky (1988, 74), "all higher mental functions are internalized social relationships". The biological subjects participate in historical events and interweave into the origins of Geometry [21].

We begin with the perceptual categories of verticality, horizontality and similarity (: recognition of the shapes and the angles). Humans turned to the objective determination of these categories by using numerical relationships and created archetypal results. In the Greek period of mathematics, an entirely new context for objectification was set up, based on Logic and the requirements of the anti-visual and the anti-empirical. Thus the ideal concepts (: prototypes) of point, line, plane, etc were invented and proofs were achieved.

Perceptual categories are transformed into conceptual formation, Tall & al (2000a). In Geometry, this modification is expressed in terms of the notions of perpendicular, parallel, and similar. In order to describe the archetypal results in this new environment, new tools are needed and the basic pattern of the triangle is created. Finally, the Pythagorean Theorem and theorems relevant to similarity were proved in Euclid's Elements, constituted the main axes of Geometry, contributed to the logical construction of space.

We summarize the previous presentation in Table 1, specifying the terminology and indicating the main steps of the program.

Table 1

Notes

[1] Similar questions and answers occurred in Lakoff (1987).

[2] Our civilization is geometric. The expression comes from Freudental (Beth - Piaget, p. 222).

[3] "the phenomenal orientation of the form is determined by directions in environment. These directions are supplied by the pull of gravity, the visual frame of reference, or instructions", Zuzne L. (1970), Visual Perception of Form, Academic Press, p. 301.

[4] "All effects of constancy, including that of Gestalt, are based on the single function of extricating the essential factor by abstracting from the inessential sensory data. The differentiation of this function attains an amazing development in service of shape constancy, and it needs only to be driven one little step further to make possible an absolutely new operation miraculously analogous to the formation of abstract, generic concepts. Not only the small children, but also higher birds and mammals, are able to perceive a supra-individual, generic Gestalt in all the individual objects of the same kind. I hold that Gestalt perception of this type is identical with that mysterious function which is generally called "Intuition", and which is generally called cognitive faculties of man. When the scientist, confronted with a multitude of irregular and apparently irreconcilable facts, suddenly 'sees' the general regularity ruling them all, when the explanation of the hitherto inexplicable all 'at once' jumps out at him with the suddenness of the revelation, the experience of this happening is fundamentally similar to that other when the hidden Gestalt in a puzzle-picture surprisingly starts out from the confusing back-ground of irrelevant detail. The German expression in die Augen springen (to spring to the eyes) is very descriptive of this progress", Smith M., (1966) Spatial Ability, University of London Press, p. 216). See also, Lehar S. (1999), Gestalt Isomorphism and the quantification of Spatial Perception, Gestalt Theory, p. 133.

[5] This perhaps constitutes the first step towards the mathematization of human experience, according to the paradigm of Pythagorean theory of music that concerns the echo's shapes (the first successful non-trivial reduction of quality to quantity obtained, Koestler A. (1968), *The Sleepwalkers*, Hutchinson of London.).

"Consider a set of elements ABC. The child may classify them according to their qualitative resemblance, for example, color, size and shape. In order that these classificatory relationships be translated into numerical ones, the child has to abstract from these qualities, so that two elements are treated at the same time as being equivalent, and as being different, that is as standing in serial relationships.", W. Mays (in Piaget, 1972, p. 5).

[6] Kneale W. & Kneale M. (1962, *The development of Logic*, Oxford, p. 2) we have the confirmation "that the notion of demonstration attracted attention first in connection with geometry". A. Szabo has the opinion that the first deductive proof started with Zeno and later went to geometry.

[7] Looking at number as a philosophical term, we may view it as a specialization of the Aristotelian category of quantity. But, quantity in general is available for statements of the type 'more or less', i.e., a primary form of the awareness of quantity that we observe children (and early civilizations as well), as it is described by Piaget (1969) and recently Stavy R. & Tirosh D. (1996), *Intuitive rules in science and mathematics: the case of "more of A - more of B"* *Int. J. Sci. Educ.* 18 (6), 653-667.

[8] "It should fit copies of itself, it should fit into the gravitational field, it should fit into the human hand...From these considerations we can derive the norm for bricks: A brick must have parallel plane sides, these pairs being orthogonal to each other. Thus the concept of the Quader has been generated by ideation, involving, however, the concepts of the 'plane', 'parallel', and 'orthogonal'...Thousands of years of practice have proved this form of brick to be most expedient one for the purpose of constructing walls", (Bender & Schreiber (1980, p. 61).

[9] "The idea that there is some core part of meaning that is invariant across all contexts or instances of a category offers a useful solution to this problem in principle, but in practice, cores for many words may be difficult or impossible to identify, just as were defining features...For instance, that the meaning of the word line is subtly different in each of many different contexts (e.g., 'tanding in line', 'crossing the line', 'typing a line of text' and that the variants are constructed at the time of hearing/reading the word from some core meaning of the word in the combination with the context in which it occurs", Malt (1999, p 333).

[10] "The central pre-Socratic concept of 'essence', 'ousia'

Essences Are Substances, Essences are Forms, Essences Are Paterns of Change ... the theory of essences fits together with the classical theory of categories, which goes back to Aristotle. In classical theory, a category is defined by a set of necessary and sufficient conditions: a list of inherent properties that each member has. A definition is a list of properties that are necessary and sufficient for something to be the kind of thing it is.... Euclid brought the folk theory of essences into mathematics in a big way. He claimed that only five postulates characterized the essence of plane geometry as a subject matter.

He believed that from this essense all other geometric truths could be derived by deduction - by reason alone! From this came the idea that every subject matter in mathematics could

be characterized in terms of an essence - a short list of axioms, taken as truths, from which all other truths about the subject matter could be deduced.”, Lakoff & Nunez, (2000, pp. 107-109).

[11] The effort of Hjelmslev (1923) is interesting, producing a natural geometry where we have no ideal notions of line or point (R. S. Tragesser (1984), *Husserl and Realism in Logic and Mathematics*, Cambridge, p. 98). In this case the results are such that the circle and its tangent have in common not a point but a small arc.

[12] Definition is not a matter of giving some fixed set of necessary and sufficient conditions for the application of a concept; instead, concepts are defined by prototypes and by types of relations of prototypes. Rather than being rigidly defined, concepts arising from our experience are open-ended, Lakoff & Johnson (p. 125)

[13] The use of the terminology 'similar' for angles is common (according to Proclus) in Thales (Heath *History of Greek Mathematics*, Ch. 4, 4.b) and is alive even in the definitions of solid geometry (Euclid's *Elements* XI, definition 10).

[14] Today in Mathematics the notions of the angles and their measurement is a deep topological result, (Dieudonné appendix II).

[15] They are special cells in the perception of angles. The recognition of the angles is a sort of innate cognitive apparatus, as evidence see in R. N. Haber & M. Hershenov (1974), *The psychology of Visual Perception*, Holt, Rinehart and Winston, London, New York. " ... cells have been found which respond to the angles between two lines, rather than to the lines alone", (p. 55) and about the infant's perception of angles (p. 358). See also in Wenderoth P. & D. White (1979), Angle - matching illusions and perceived orientation, *Perception* Vol. 8, pp. 565-575.

[16] A detailed analysis of Plato's geometric ideas, in connection with the above situations, can be found in Popper's writings in particular see his essay "Plato and Geometry" pp. 251-270 in K. Popper (1988), *The world of Parmenides*, Routledge.

[17] After the discovering of 'Non-Euclidean Geometry', the Theory of Similarity constitutes a characterization of 'Euclidean Geometry', since it is proved that the relations of 'similarity' and 'congruence' are distinct if and only if geometry is Euclidean.

[18] There is an extensive bibliography for n-dimensional extensions of the Pythagorean Theorem. All this and recent results can be found in the article of J. P. Quadrat & J. B. Lassere & J. B. Hiriart - Urruty, Pythagora's Theorem for Areas, in the *American Mathematical Monthly*, 108, 2002.

[19] The proofs of these facts were carried out through the development of Hyperbolic Geometry and its models. Thus these are 'metatheorems' in Elementary Geometry. However, there are proofs which do not refer directly to the models, see for instance, Millman and Parker: *Modern Geometry, A Metric approach with Models*, Springer-Verlag, 1981, pp. 219 -227.

[20] D. Laugwitz (1999), "B. Riemann, Turning point in the conception of Mathematics", Birkhäuser, Boston.

[21] "Geometry and the science most closely related to it have to do with space-time and the shapes, figures, also shapes of motions, alternations of deformation, etc., that are possible with space-time, particularly as measurable magnitudes. It is now clear that even if we know

almost nothing about the historical surrounding world of the first geometers, this much is certain as invariant, essential structure: that it was a world of 'things' (including the human beings themselves as subjects of this world); that all things necessarily had to have a bodily character - although not all things could be mere bodies, since the necessarily coexisting human beings are not thinkable as mere bodies and, like even cultural objects which belong with them structurally, are not exhausted in corporeal being," E. Husserl (1999, p 375).

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